Ch. 7 The Wave Nature of Particles

References:

1. Young & Freedman, “University Physics”, 13th ed. Ch. 38, 39

2. Halliday et al., “Principles of Physics”, 9th ed. Ch. 38, 39

Outline

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7.2 Electrons and Matter Waves
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7.1 De Broglie Waves

Light behaves like wave in some situations and in others like particles. If nature is symmertic, the duality should also hold for matter.
In 1924, de Broglie postulated that matter particles also behave like waves in some situations and the following relations hold:

$$\lambda = \frac{h}{p} \quad \text{and} \quad E = h\,f$$

where E = energy; p = momentum

Louis de Broglie
(1892-1987)

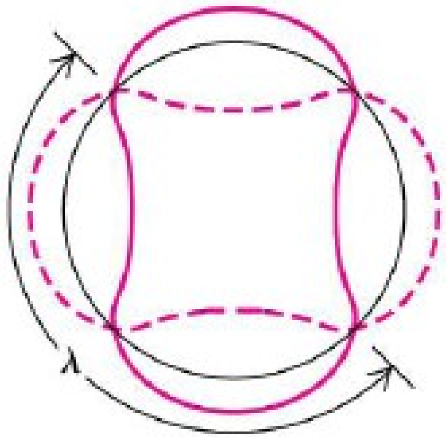
• ***The Bohr Model and de Broglie Waves***

Consider an electron as a standing wave fitted around a circle in one of the Bohr orbits

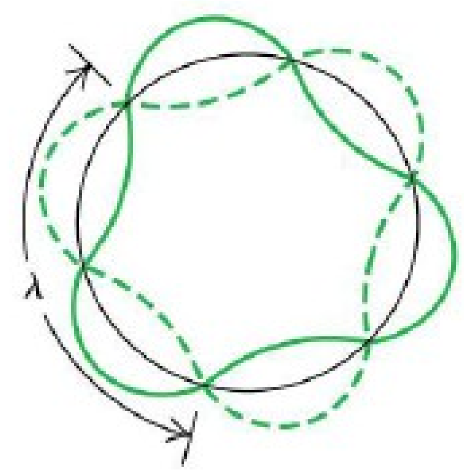
$$2\pi r = n\lambda \qquad n = 1, 2, 3, \ldots$$

For non-relativistic particle: $\lambda = \dfrac{h}{p} = \dfrac{h}{mv}$

=> $$L = mvr = n\frac{h}{2\pi}$$



$n = 2$

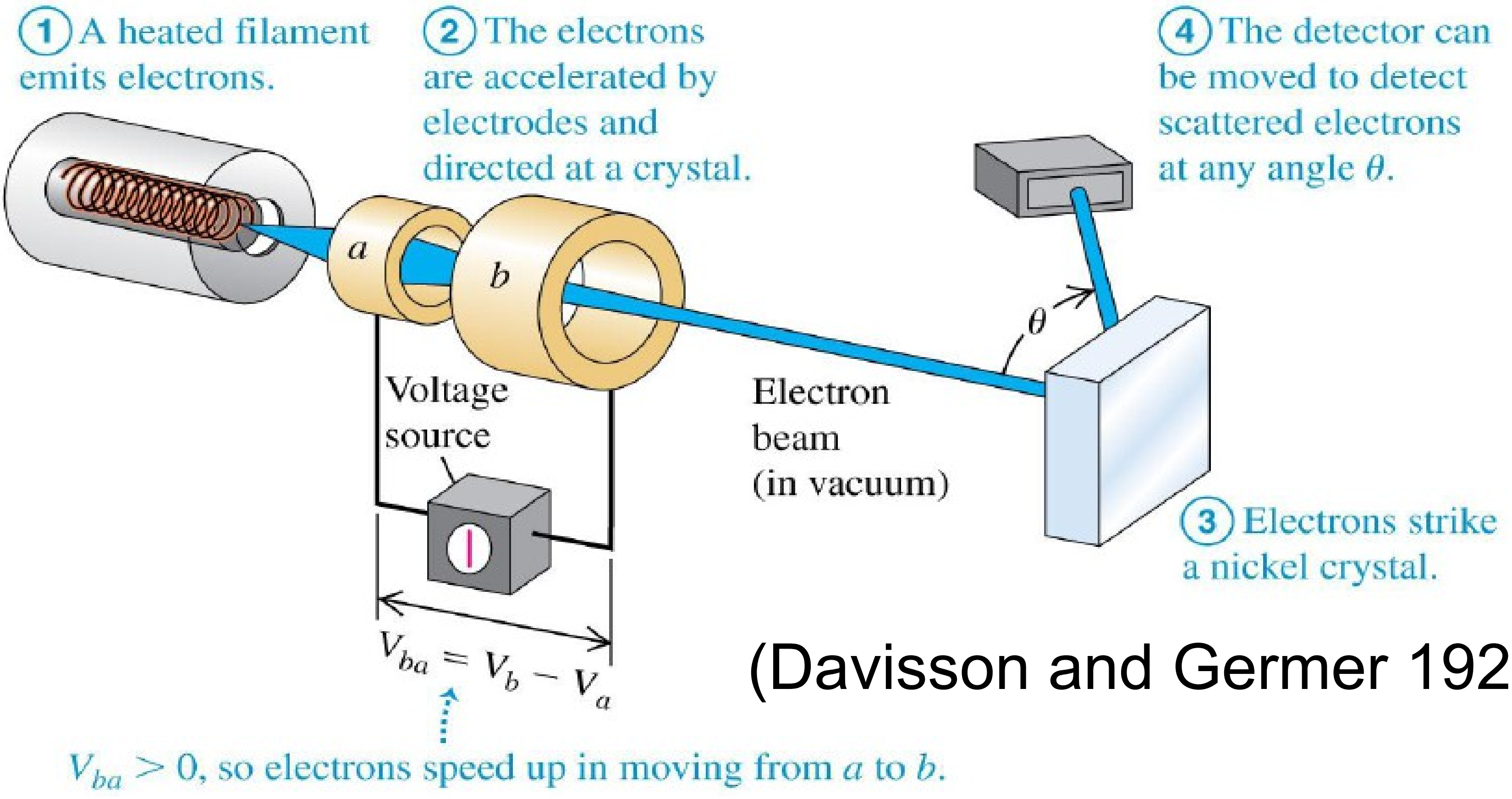


$n = 3$

This is Bohr's result that the angular momentum of the electron is quantized.

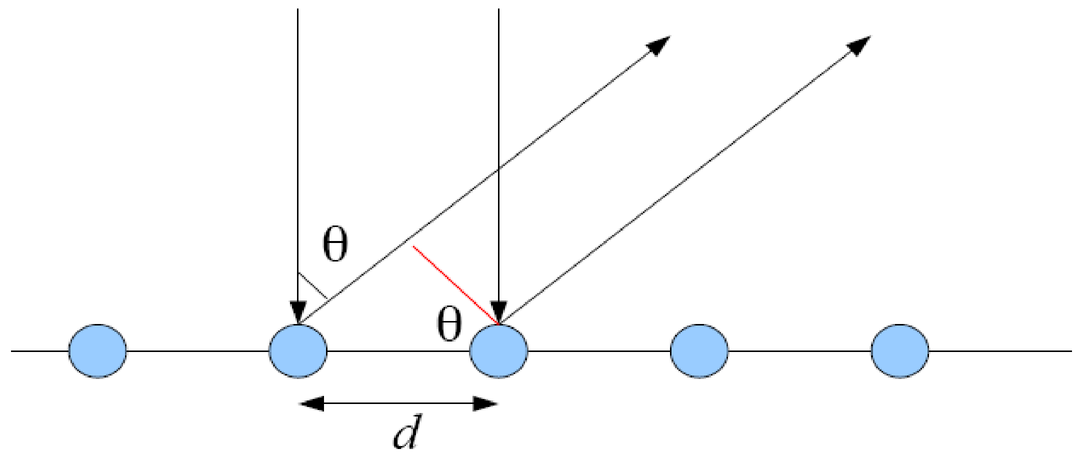
7.2 Electrons and Matter Waves

- ***Electron Diffraction***



(Davisson and Germer 1927)

Consider a beam of electrons scattered by a target crystal. The electrons were scattered primarily by the planes of atoms at the surface



Wave nature of electrons => the plane acts as a reflecting diffraction grating

Max reflection intensity occurs at

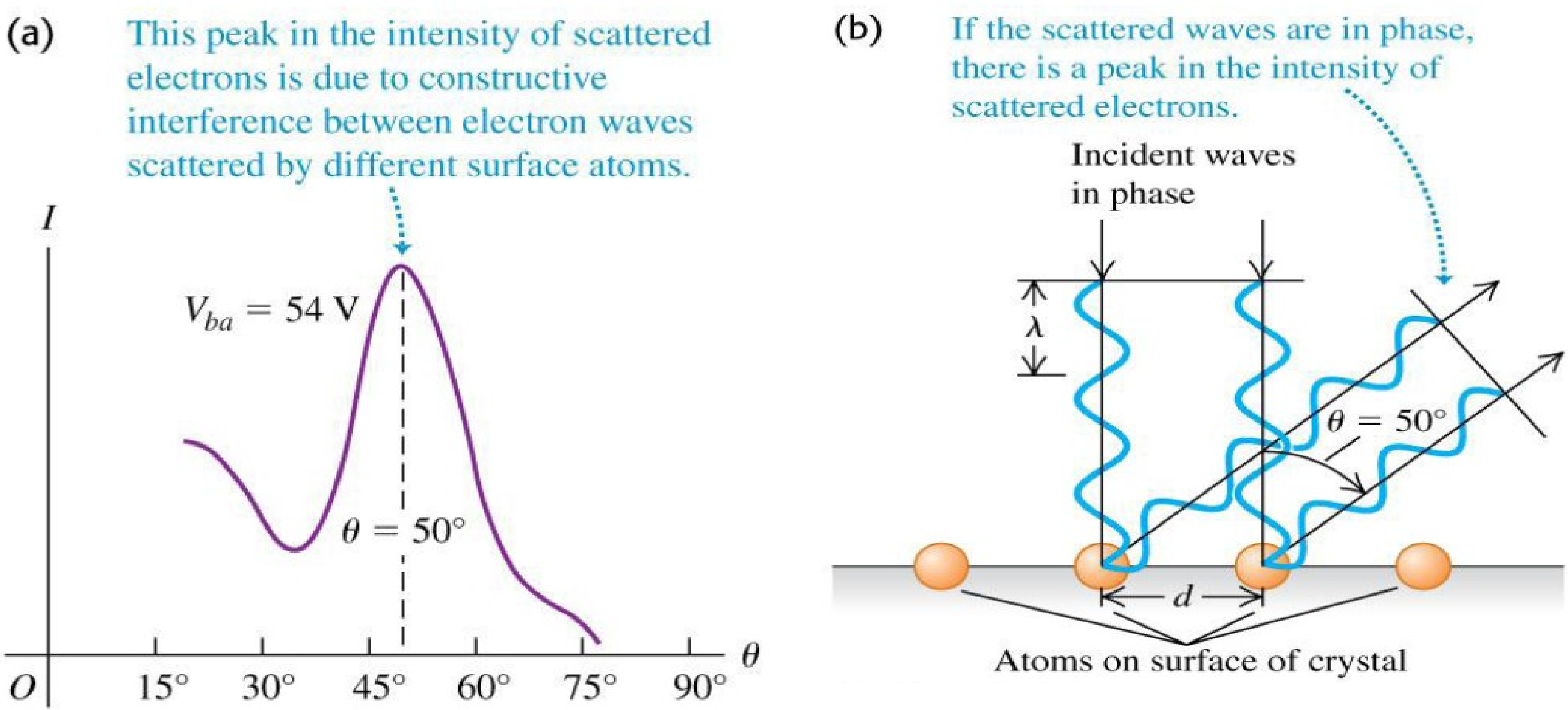
$$d \sin\theta = m\lambda \qquad (m = 1, 2, 3, \ldots)$$

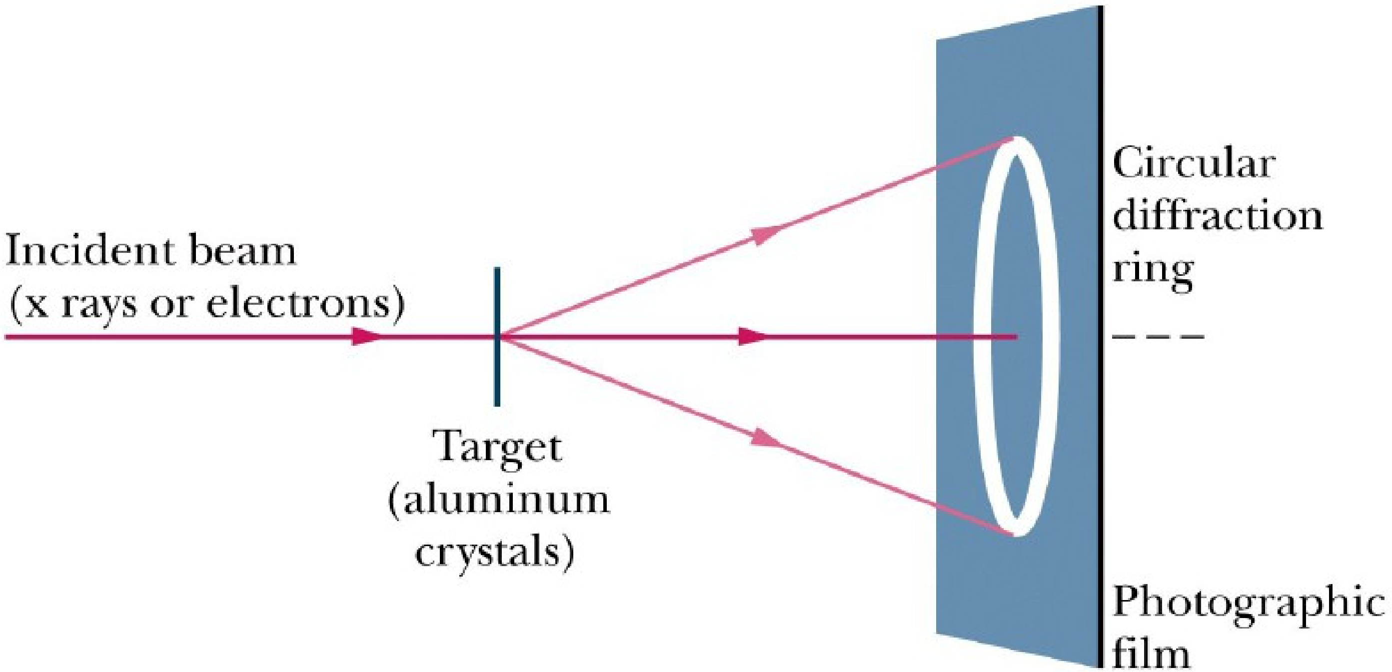
(ie, when the scattered waves are in phase)

Consider an electron freely accelerated through a potential V_{ab} :

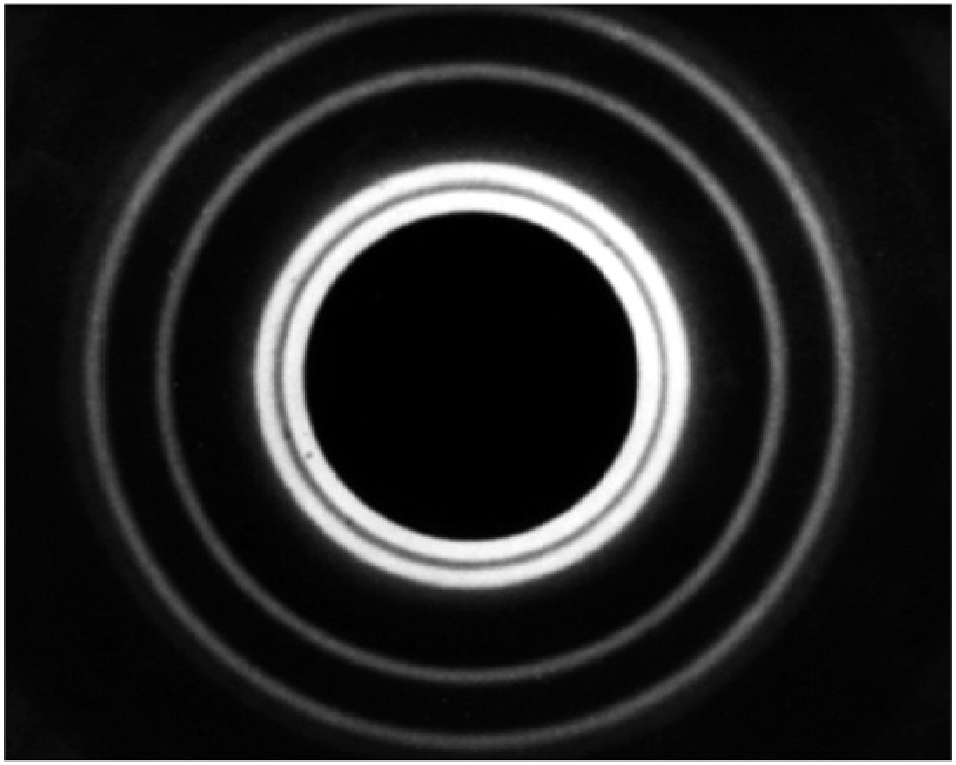
$$eV_{ab} = \frac{p^2}{2m}$$

The de Broglie wavelength is: $\lambda = \frac{h}{p} = \frac{h}{\sqrt{2meV_{ab}}}$

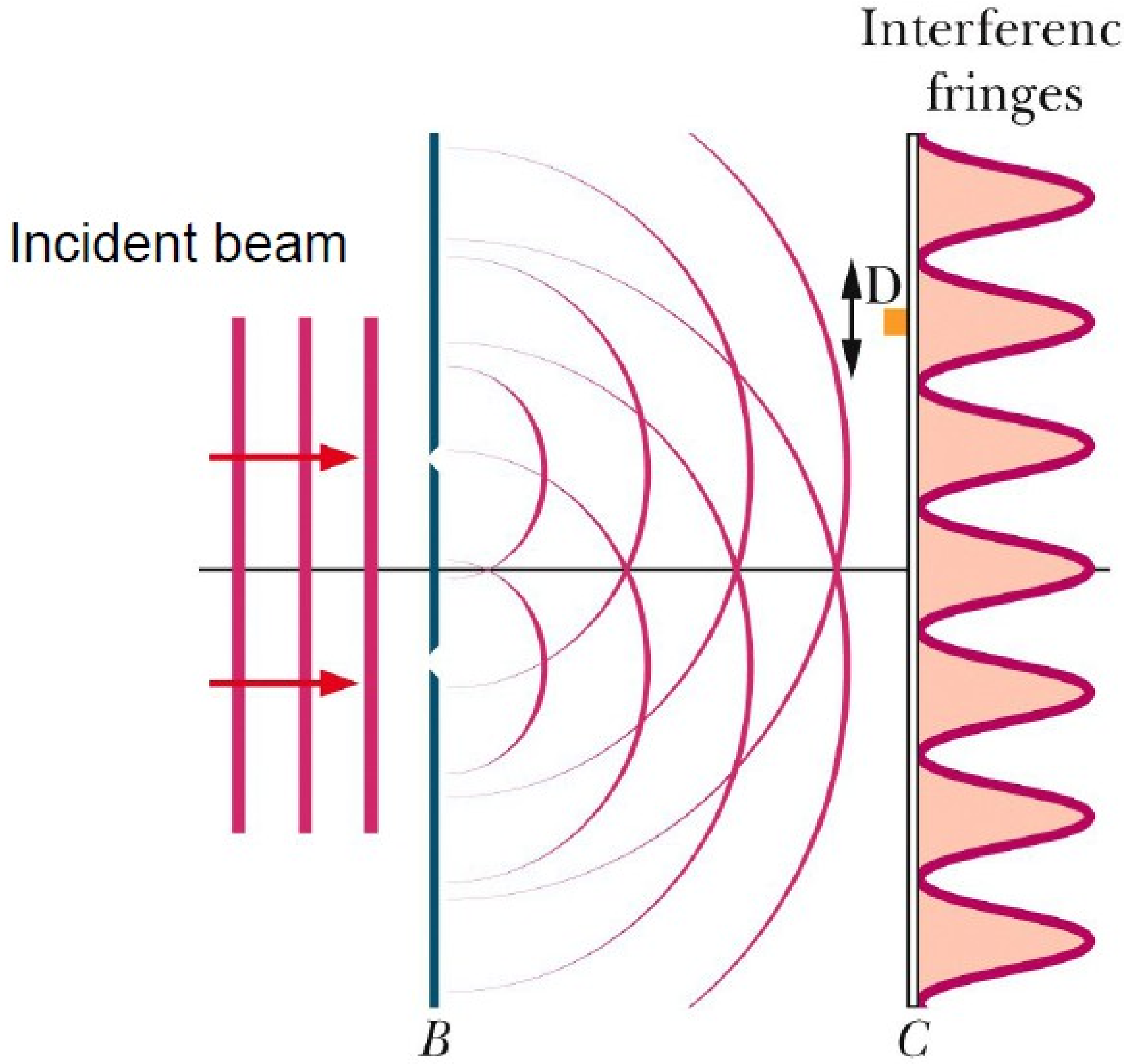




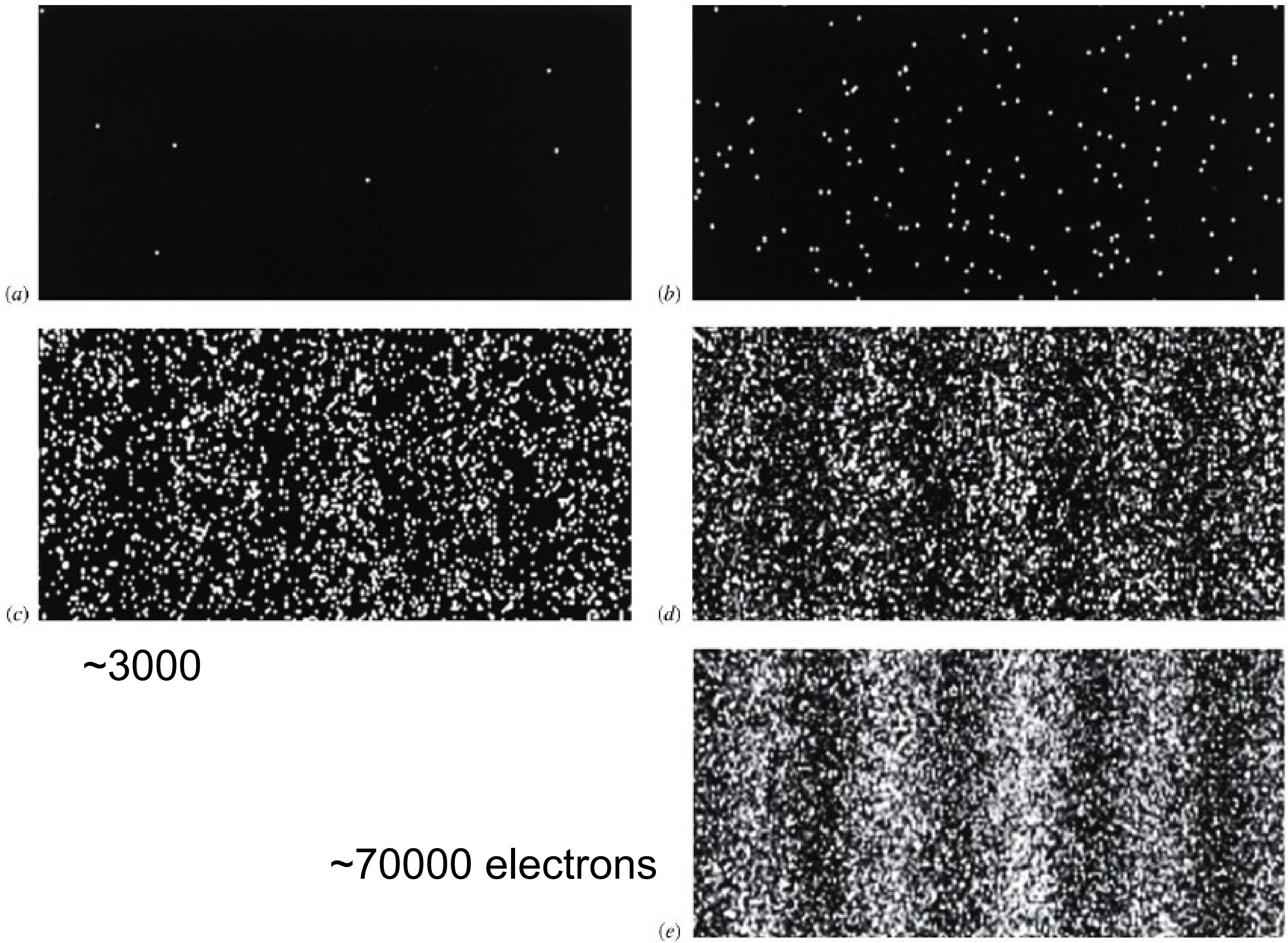
Diffraction pattern
by an electron beam



- ***Electron Double-Slit Interference***



Photographs showing the buildup of an interference pattern by a beam of electrons:



7.3 Wave Functions and the Schrödinger Equation

In atomic or subatomic scale, we use a wave function to describe the state of a particle:

$$\Psi\left(x\,,\,y\,,\,z\,,\,t\right)$$

The wave function for a particle contains all the information that can be known about the particle.

• *Interpretation of the Wave Function*

In 1926, Max Born proposed the following interpretation:

$$|\Psi(x, y, z, t)|^2 dV =$$ Probability of finding the particle at time t within a volume dV around the point (x,y,z)

Max Born
(1882-1970)

Remark:

1. We shall see that Ψ is complex so that $|\Psi|^2 = \Psi\,\Psi^*$

(complex conjugate)

2. The particle must be somewhere in the universe. The wave function must be normalized:

$$\boxed{\int_{\text{all space}} |\Psi|^2\, dV = 1}$$

3. $|\Psi|^2$ is the probability density (probability distribution function).

- ***Stationary States***

In general, $|\Psi(x,y,z,t)|^2$ at a particular point varies with time.

If the particle is in a state of definite energy, $|\Psi|^2$ at each point is independent of time (stationary state).

The wave function for any state can be expressed as a combination of stationary states.

For a particle in a state of definite energy E (stationary state):

$$\boxed{\Psi(x, y, z, t) = \psi(x, y, z)\, e^{-iEt/\hbar}}$$

Note: $$\begin{aligned} |\Psi(x, y, z, t)|^2 &= \Psi(x, y, z, t)\,\Psi^*(x.y, z, t) \\ &= \psi(x, y, z)\, e^{-iEt/\hbar}\, \psi^*(x, y, z)\, e^{+iEt/\hbar} \\ &= |\psi(x, y, z)|^2 \end{aligned}$$

For a stationary state, the probability distribution function does not depend on time

- ***The Time-Independent Schrödinger Equation***

How to determine $\psi(x,y,z)$ and E ?

They are determined by solving the Schrödinger equation in QM

For a particle of mass m moving in one dimension:

$$\boxed{-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2}+U(x)\psi(x)=E\psi(x)}$$

(one-dimensional Schrödinger equation)

where $U(x)$ = potential energy

Remark:

1. Schrödinger equation is a physical principle. It cannot be derived from other principles!

2. We know that it is correct because its predictions agree with experiments!

3. There is also a time-dependent Schrödinger equation.
   However, we don't need that to study stationary states.

Erwin Schrödinger
(1887-1961)

• *Wave Function for a Free Particle*

Take $U(x)$ = 0:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2}=E\,\psi(x)$$

=> $$\psi''(x)=-k^2\psi(x) \qquad \left(k^2=2mE/\hbar^2\right)$$

=> $$\psi(x)=C\cos(kx)+D\sin(kx)$$

We can also write

$$\psi(x)=A\,e^{ikx}$$ (A = complex number)

For a free particle:

$$E = \frac{p^2}{2m} \quad \Rightarrow \quad k^2 = \frac{2mE}{\hbar^2} = \frac{p^2}{\hbar^2}$$

From de Broglie relations:

$$\lambda = \frac{h}{p} \quad \Rightarrow \quad k = \frac{2\pi}{\lambda}$$ (wave number)

$$E = hf \quad \Rightarrow \quad \omega = 2\pi f = \frac{E}{\hbar}$$ (angular frequency)

=> Free-particle wave function:

$$\boxed{\begin{aligned} \Psi(x,t) &= \psi(x)\, e^{-iEt/\hbar} \\ &= A\, e^{i(kx - \omega t)} \end{aligned}}$$

A free particle has a definite momentum p in the x-direction.

No uncertainty in momentum $\Delta p = 0$

$$\begin{aligned}|\Psi(x,t)|^2 &= \Psi(x,t)\,\Psi^*(x,t)\\ &= A\,e^{i(kx-\omega t)}\,A^*\,e^{-i(kx-\omega t)}\\ &= |A|^2 = \text{constant}\end{aligned}$$

=> We are equally likely to find the particle anywhere in space!

$$\Delta x = \infty$$

Note: The wave function cannot be normalized

$$\int_{-\infty}^{+\infty} |\Psi(x,t)|^2\, dx = \infty$$

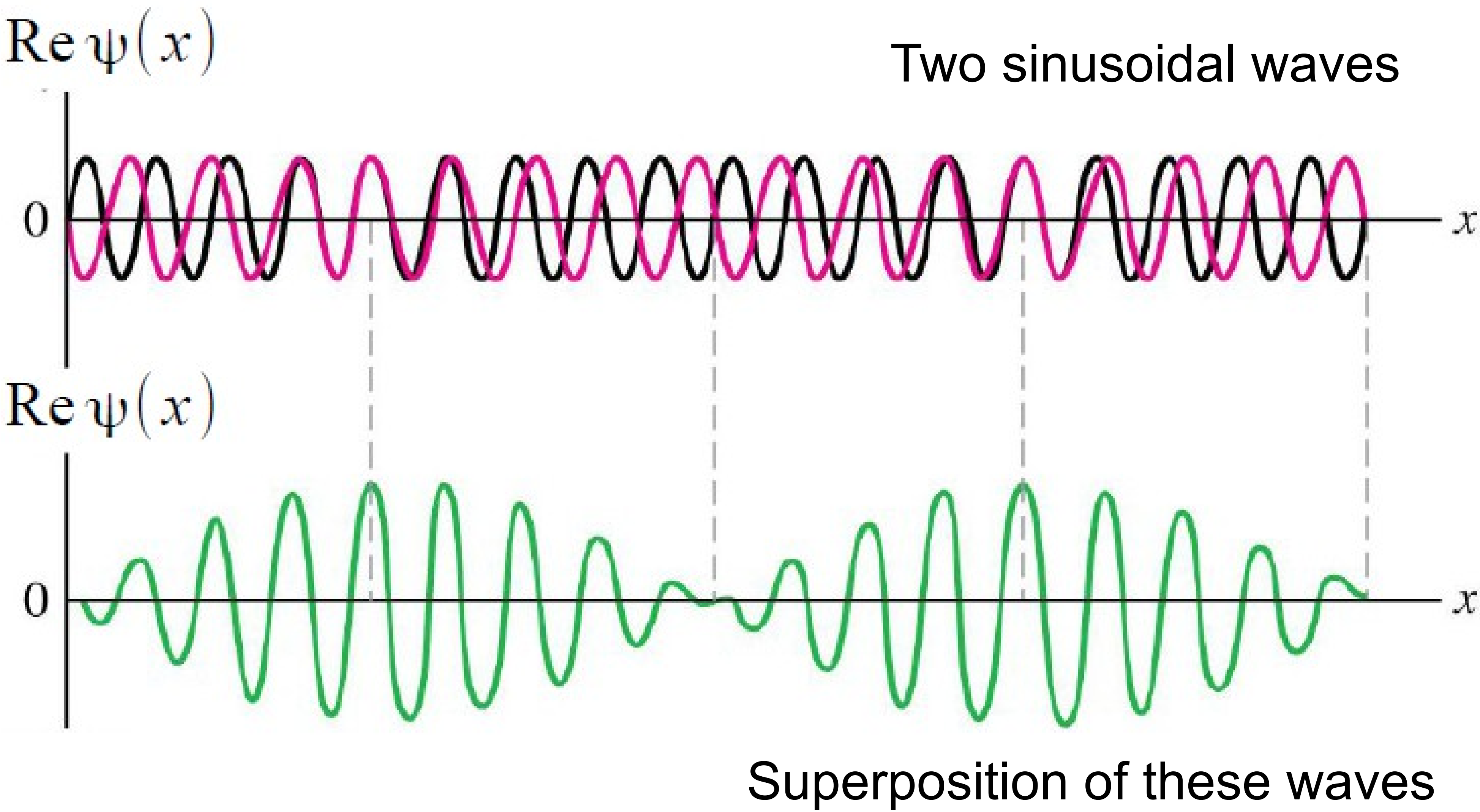
• ***Wave Packets***

In real situations, we always have some idea where a particle is. We need a wave function that is localized in space (wave packet).

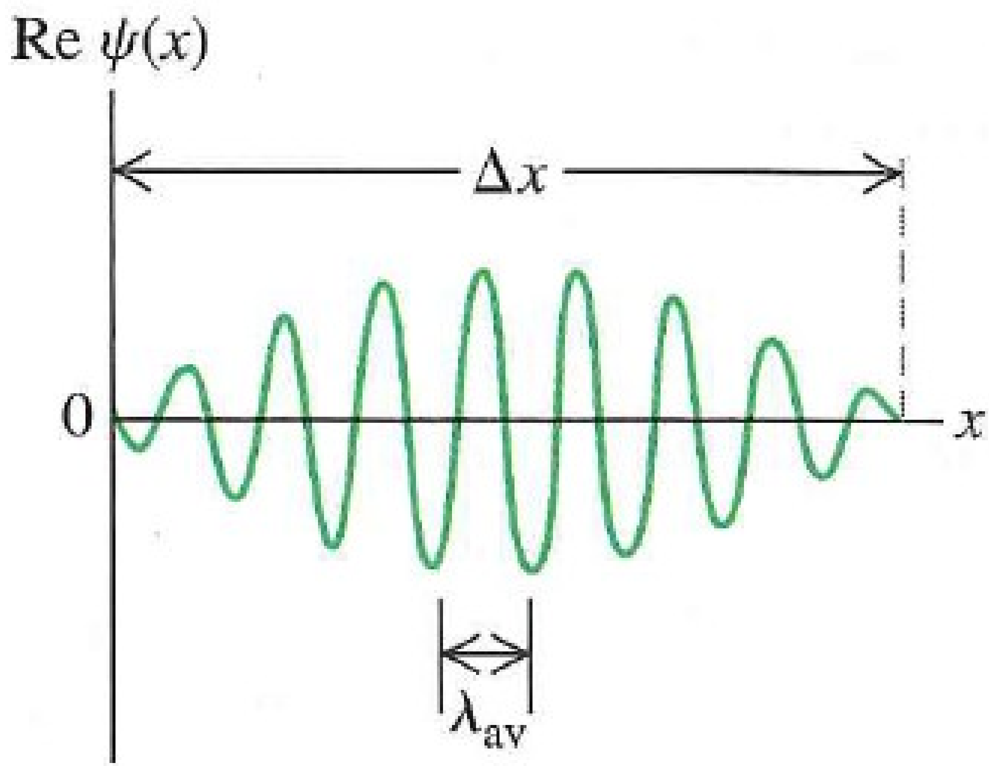
A wave packet can be constructed by superimposing two or more sinusoidal waves.

Example:
Consider two waves (with slightly different frequencies) at t = 0

$$\begin{aligned}\psi(x) &= A_1 e^{ik_1 x} + A_2 e^{ik_2 x} \\ &= \left[A_1 \cos(k_1 x) + A_2 \cos(k_2 x)\right] + i\left[A_1 \sin(k_1 x) + A_2 \sin(k_2 x)\right]\end{aligned}$$


Re ψ(x)
Two sinusoidal waves
0
x
Re ψ(x)
0
x
Superposition of these waves

Superposition of a large number of sinusoidal waves with different wave numbers (k) and appropriate amplitudes:



$$\lambda_{av} = \frac{2\pi}{k_{av}}$$

This localized pulse has aspects of both particle and wave.

Note: The particle's momentum no longer has a definite value.

$$\Delta p \neq 0$$

7.4 The Uncertainty Principle

In general, it is impossible to determine the position and momentum of an object with arbitrary great precision at the same time:

$$\boxed{\Delta x \Delta p_x \geq \hbar/2} \qquad (\hbar = h/2\pi)$$

Similarly $\Delta y \Delta p_y \geq \hbar/2$, $\Delta z \Delta p_z \geq \hbar/2$

Werner Heisenberg
(1901-1976)

Note:
The uncertainty in one coordinate is not related to the uncertainty in a different component of p.
(e.g., Δx is not related directly to Δp_y)

- ***Uncertainty in Energy***

There is also an uncertainty principle for energy. The uncertainty ΔE depends on the time interval Δt during which the system remains in the given state:

$$\boxed{\Delta E \, \Delta t \geq \hbar/2}$$

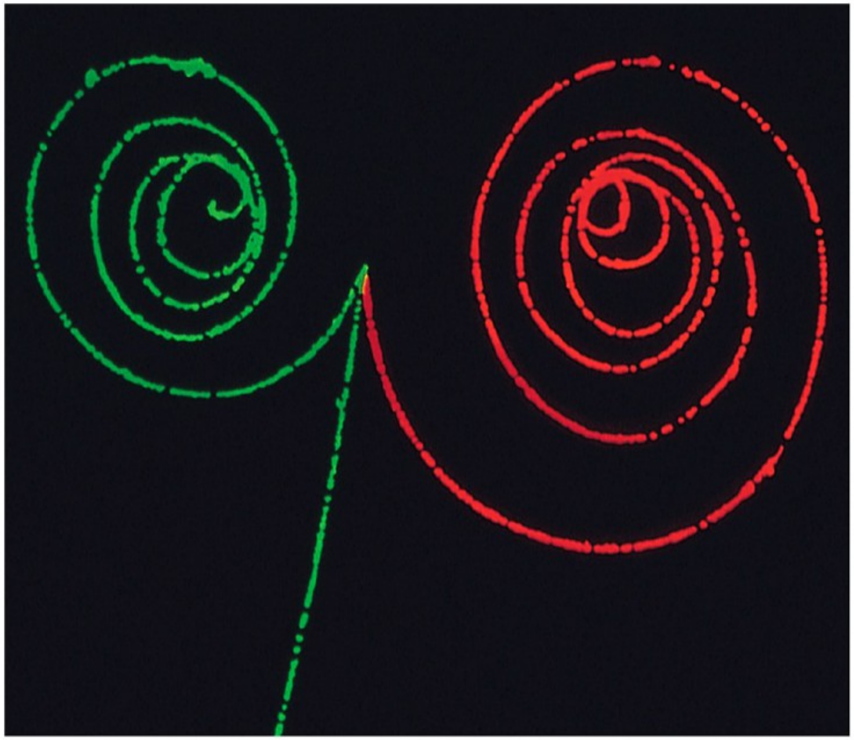
Heisenberg uncertainty principle for energy and time interval

7.5 Wave-Particle Duality

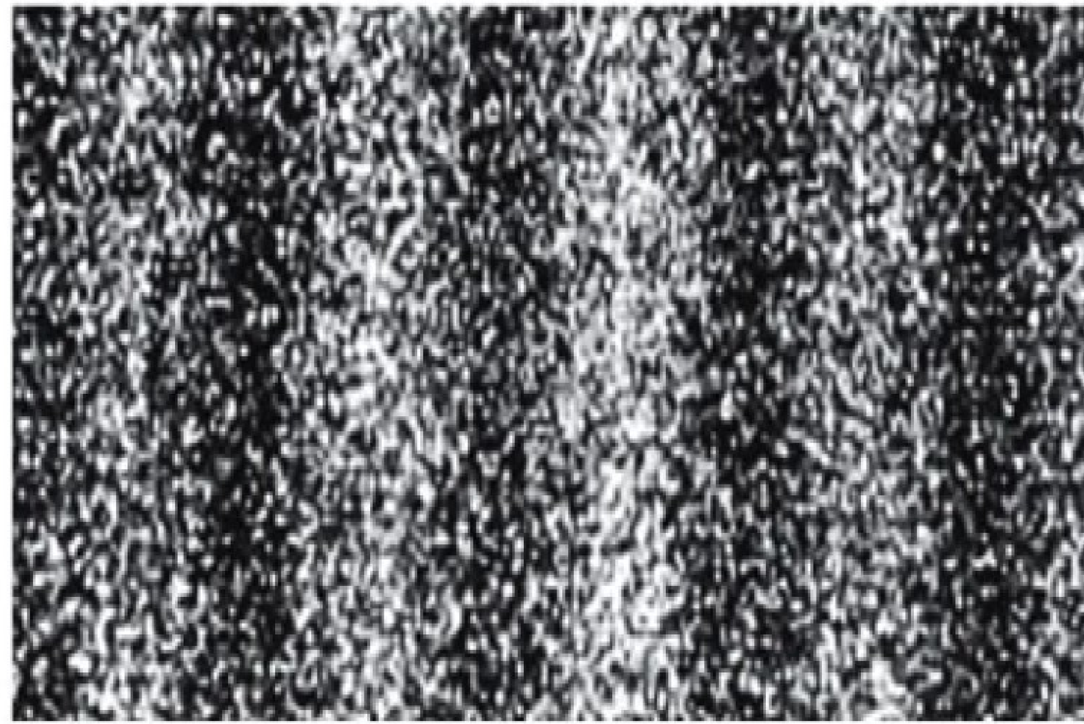
Why electrons (or photons) can be wavelike in some experiments and particlelike in others?

Principle of complementarity: (Bohr 1927)

The wave and the particle aspects of a quantum entity are both necessary for a complete description. However, the two aspects cannot be revealed simultaneously in a single experiment. The aspect that is revealed is determined by the nature of the experiment being done.

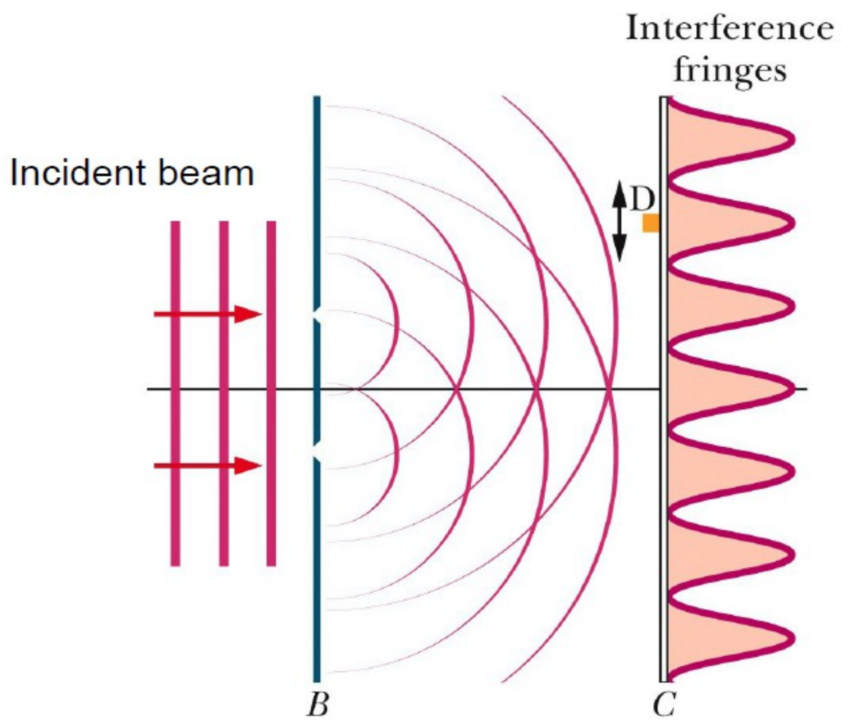


Trajectories of matter particles in a bubble chamber



Interference pattern of matter wave in a double-slit experiment

Case Study:



Which slit does an individual electron pass through?

If we put a detector in front of each slit, so that if an electron passes through it, it will generate a signal. We can then relate the signal to the "screen signal" detected by D, thus identifying the path of the electron.

What would happen if we succeed to do this?

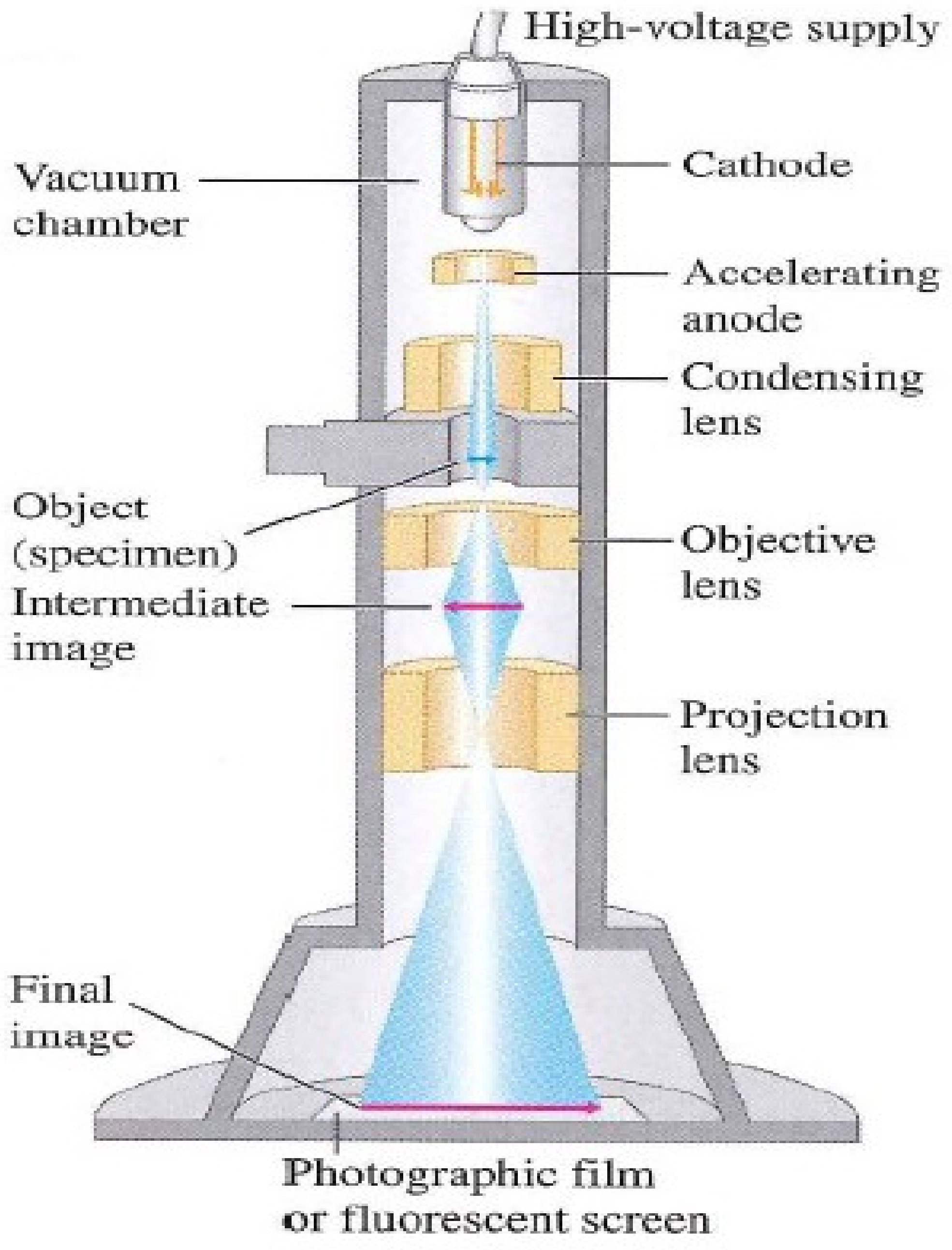
Appendix: The Electron Microscope

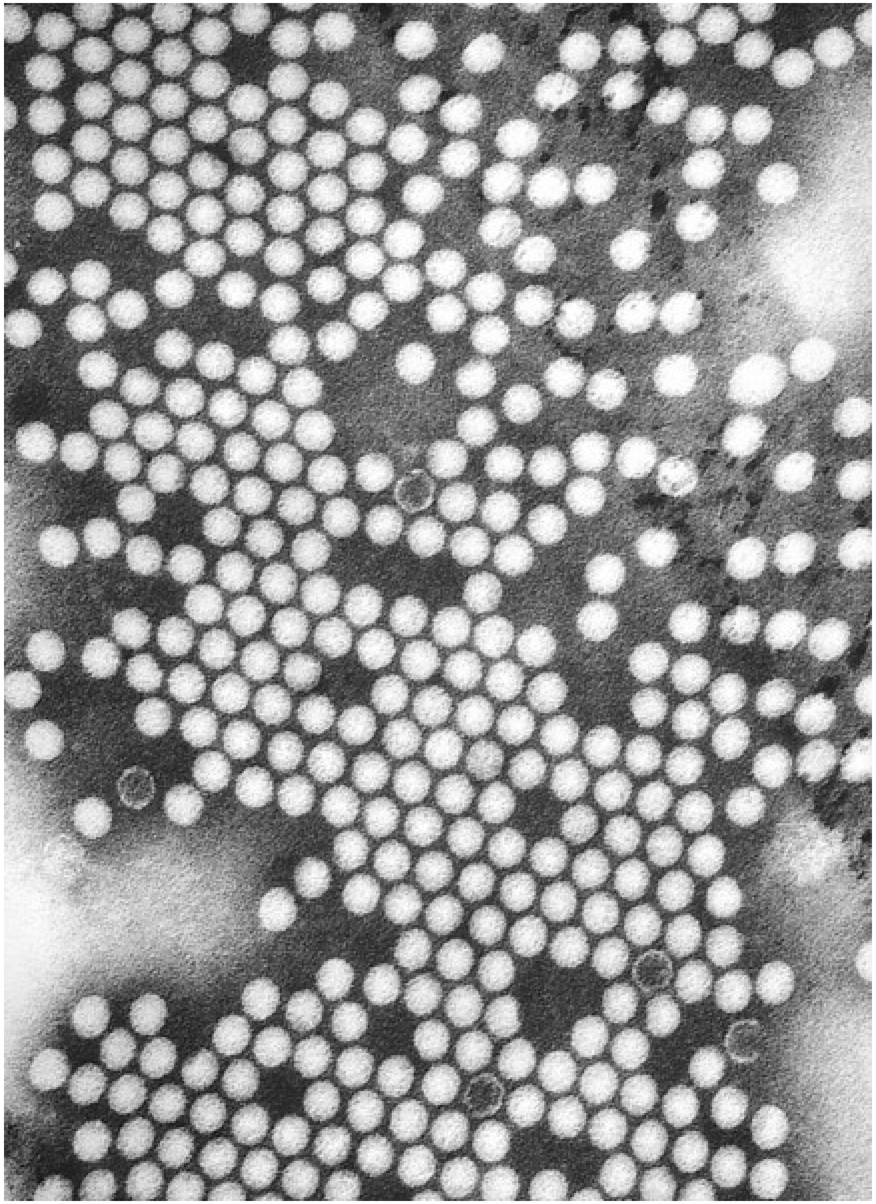
(Optical microscope) Light beam	(Electron microscope) Electron beam
1. bent by reflection or refraction	bent by E- or B-field
2. brought to convergence by a converging lens or concave mirror	brought to convergence by E- or B-field

Note: Resolution of an optical microscope is limited by diffraction effect (determined by the wavelength ~500 nm).

=> resolution of an electron microscope is much higher (electron wavelength << wavelength of visible light)

- ***Transmission Electron Microscope*** **(*TEM*)**





TEM image of the polio virus
(The virus is 30 nm in size)

• ***Scanning Tunneling Microscope*** (***STM***)